

# Combined gravitational and electromagnetic self-force on charged particles in electrovac spacetimes

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## Abstract

We consider the self-force on a charged particle moving in a curved spacetime with a background electromagnetic field, extending previous studies to situations in which gravitational and electromagnetic perturbations are comparable. The formal expression  $f_\alpha^{ret}$  for the self-force on a particle, written in terms of the retarded perturbed fields, is divergent, and a renormalization is needed to find the particle's acceleration at linear order in its mass  $m$  and charge  $e$ . We assume that, as in previous work in a Lorenz gauge, the renormalization for accelerated motion comprises an angle average and mass renormalization. Using the short distance expansion of the perturbed electromagnetic and gravitational fields, we show that the renormalization is equivalent to that obtained from a mode sum regularization in which one subtracts from the expression for the self-force in terms of the retarded fields a singular part field comprising only the leading and subleading terms in the mode sum. The most striking part of our result, arising from a remarkable cancellation, is that the renormalization involves no mixing of electromagnetic and gravitational fields. In particular, the renormalized mass is obtained by subtracting (1) the purely electromagnetic contribution from a point charge moving along an accelerated trajectory and (2) the purely gravitational contribution from a point mass moving along the same trajectory. In a mode-sum regularization, the same cancellation implies that the required regularization parameters are sums of their purely electromagnetic and gravitational values.

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## I. INTRODUCTION

The last fifteen years have seen a number of papers dealing with the self-force on a small mass  $m$ , modeled as a particle moving on a background spacetime, and on the analogous problem for a charge  $e$  or scalar charge  $q$  (see reviews by Barack [1] and Poisson *et al.* [2]). The work on motion of a small mass has been restricted to uncharged particles whose motion at zeroth order in  $m$  is a geodesic of an unperturbed vacuum spacetime; corresponding work on charged particles has assumed a negligible contribution to the self-force from the perturbed gravitational field. The primary motivation has been to study extreme mass-ratio inspiral (EMRI), the inspiral of stellar-size black holes or neutron stars orbiting supermassive galactic black holes, with the electromagnetic and scalar studies serving as toy problems. There has, however, been recent interest in whether self-force plays a fundamental role in enforcing cosmic censorship by preventing one from overcharging (or overspinning) a near-extreme black hole [3–6]. In this context, one would like to analyze scenarios in which gravitational and electromagnetic perturbations have comparable magnitude.

A charged particle moving on a smooth background spacetime  $M, g_{\alpha\beta}$  with electromagnetic field  $F_{\alpha\beta}$  has trajectory  $z(\tau)$  satisfying the Lorentz force law

$$ma_\alpha = eF_{\alpha\beta}u^\beta, \quad (1)$$

For a smooth perturbation  $g_{\alpha\beta} + h_{\alpha\beta}$ ,  $F_{\alpha\beta} + \delta F_{\alpha\beta}$  of the geometry and electromagnetic field, the 4-velocity  $\bar{u}^\alpha$  of the perturbed trajectory satisfies

$$m\bar{u}^\beta\nabla_\beta\bar{u}_\alpha - eF_{\alpha\beta}\bar{u}^\beta = \delta F_{\alpha\beta}u^\beta - q_\alpha^\delta(\nabla_\beta h_{\gamma\delta} - \frac{1}{2}\nabla_\delta h_{\beta\gamma})u^\beta u^\gamma := f_\alpha = f_\alpha^{EM} + f_\alpha^{GR}, \quad (2)$$

with  $\bar{u}^\alpha$  normalized by the background metric,

$$g_{\alpha\beta}\bar{u}^\alpha\bar{u}^\beta = -1. \quad (3)$$

When the perturbation is due to the retarded fields of the particle itself, the formal expression (2) for the self-force, with  $f_\alpha^{ret} = f_\alpha[h^{ret}, \delta F^{ret}]$ , diverges at the particle. Mino *et al.* [7] and Quinn and Wald [8] obtained equivalent prescriptions for renormalizing  $f_\alpha^{ret}$ , following work by DeWitt and Brehme [9] (corrected by Hobbs [10]) on a charged particle moving in a vacuum background spacetime.<sup>1</sup>

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<sup>1</sup> The most recent and rigorous work justifying this MiSaTaQuWa renormalization uses variants of matched

When the unperturbed motion is geodesic, the renormalized self-force at a point  $z$  of the particle's trajectory can be obtained as the  $\rho \rightarrow 0$  limit of an angle average of  $f_\alpha^{ret}$  over a sphere  $S_\rho$  of geodesic distance  $\rho$  from  $z$  [14]. Explicitly,

$$f_\alpha^{ren}(z) = \lim_{\rho \rightarrow 0} \langle f_\alpha^{ret} \rangle_\rho = \lim_{\rho \rightarrow 0} \int_{S_\rho} d\Omega f_\alpha^{ret}, \quad (4)$$

where the components  $f_\alpha^{ret}$  are given in Riemann normal coordinates (RNCs) centered at  $z$ . (Equivalently, the average is taken in the tangent space at  $z$  with  $f_\alpha^{ret}$  pulled back by the exponential map.) When the trajectory is accelerated, we show in a previous paper [15] (henceforth *Paper I*) that the angle average leaves a term proportional to  $a_\alpha/\rho$ , which can be regarded as a renormalization of the mass. The renormalized self-force on an electromagnetic or scalar charge moving on an accelerated trajectory has the form

$$f_\alpha^{ren} = \lim_{\rho \rightarrow 0} \langle f_\alpha^{ret} \rangle_\rho - m^{sing}(\rho) a_\alpha, \quad (5)$$

with  $m^{sing}(\rho) \propto \rho^{-1}$ . For the more general situation we consider here, with electromagnetic and gravitational perturbations each contributing to the self-force, we again assume that  $f_\alpha^{ren}$  is given by Eq. (5).

We find that  $m^{sing}$  is a sum  $m^{sing} = m^{GR} + m^{EM}$  of gravitational and electromagnetic parts.  $m^{GR}$  is proportional to  $m^2$  and comes solely from the gravitational perturbation associated with the accelerated motion of a mass  $m$ ;  $m^{EM}$  is proportional to  $e^2$  and comes solely from the electromagnetic perturbation associated with the motion of a charge  $e$ . Contributions to  $f_\alpha^{sing}$  from terms proportional to  $em$  are equal and opposite and give no net mixed contribution to  $m^{sing}$ .

We show that this renormalization is equivalent to performing an angle average after subtracting a singular part of  $f_\alpha^{ret}$  associated with the short-distance expansion of the perturbations  $h_{\alpha\beta}$  and  $\delta A_\alpha$  to subleading order in  $\epsilon$ . An independent calculation by Zimmerman and Poisson finds the Detweiler-Whiting forms of the singular fields  $h_{\alpha\beta}^S$  and  $\delta^S A_\alpha$  and the corresponding form  $f_\alpha^S$  of the singular part of the expression for the self force, and our renormalizations agree.

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asymptotic expansion to find the motion of small bodies in the limit where the size of the body and its mass (or charge) simultaneously shrink to zero. See Gralla *et al.* [11, 12] (with a formal proof for an electromagnetic charge), Pound [13], and Poisson *et al.* [2], who also review the history and give a comprehensive bibliography.

Most explicit calculations of the self-force on particles moving in Kerr or Schwarzschild geometries, however, have used a mode-sum form of the renormalization introduced by Barack and Ori [16, 17], with early development and first applications by them, by Mino *et al.*, and by Burko [18–20] (see Refs. [1, 2] for reviews of later work). Here one expresses  $f_\alpha^{sing}$  and  $f_\alpha^{ret}$  as sums of angular harmonics on a sphere through the particle, writing the renormalized self-force as the convergent sum  $\sum_{\ell=0}^{\infty} (f_\alpha^{ret,\ell} - f_\alpha^{sing,\ell})$ , with  $f_\alpha^{ret,\ell}$  and  $f_\alpha^{sing,\ell}$  each a sum over  $m$  of its  $\ell, m$  harmonics. Paper I extended to accelerated motion a fundamental feature of this mode sum in a Lorenz or smoothly related gauge: Only the leading and subleading terms in  $\ell^{-1}$  give nonzero contributions to the singular expression for the self-force. For a point particle with scalar or electromagnetic charge, and for a point mass,  $f_\alpha^{sing,\ell}$  has the form

$$f_\alpha^{sing,\ell\pm} = \pm A_\alpha L + B_\alpha, \quad (6)$$

where  $L = \ell + 1/2$ ,  $A_\alpha$  and  $B_\alpha$  are independent of  $\ell$ , and the sign  $\pm$  refers to a limit of the direction-dependent singular expression taken as one approaches the sphere through the particle from the outside or inside. We show here that the same form holds for a charged particle moving in an electrovac spacetime. The coupling of electromagnetic and gravitational perturbations in the Einstein-Maxwell system means that the renormalized self-force has mixed contributions proportional to  $em$ , but no such mixed terms arise in the renormalization: The cancellation mentioned above implies that *the regularization parameters  $A_\alpha$  and  $B_\alpha$  are just the sums of their values for purely gravitational and purely electromagnetic contributions* to the terms arising in mode-sum regularization of an accelerated particle  $e, m$ . (Higher order regularization parameters are useful for convergence, and they presumably do involve mixed terms, but they multiply vanishing sums.)

The plan of the paper is as follows. In Sect. II, we find explicit expansions in RNCs for the electromagnetic and gravitational perturbations produced by a particle of mass  $m$  and charge  $e$  moving in an electrovac background spacetime. These expansions can be identified with the singular parts  $h_{\alpha\beta}^{sing}$  and  $\delta A_\alpha$  of the perturbations and provide an expression for the  $f_\alpha^{sing}$ , again to subleading order. We obtain the cancellation of mixed gravitational and electromagnetic terms (terms proportional to  $em$ ) mentioned above, and write the explicit expression for the mass renormalization. In Sect. (III), we obtain the regularization parameters  $A_\alpha$  and  $B_\alpha$  required for mode-sum regularization of the self-force. They are

the simply the sum of terms previously obtained in Paper I for accelerated charges and accelerated masses, respectively. Finally, using forms for the perturbed gravitational and scalar field of particle with scalar charge moving in a scalarvac spacetime provided by Poisson and Zimmerman [21], we show that in the renormalization of the self-force, the contributions of gravitational and electromagnetic fields are similarly decoupled.

## II. SELF-FORCE IN ELECTROVAC SPACETIMES

We consider a point particle of mass  $m$  and charge  $e$  moving with trajectory  $z(\tau)$  in a smooth electrovac spacetime,  $(M, g_{\alpha\beta}, F_{\alpha\beta})$ , with  $F_{\alpha\beta}$  a sourcefree electromagnetic field. The metric  $g_{\alpha\beta}$  of the background spacetime then has as its source the stress-energy tensor of  $F_{\alpha\beta}$ ,

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} = 2 \left( F_{\alpha\mu} F_{\beta}{}^{\mu} - \frac{1}{4} g_{\alpha\beta} F^{\mu\nu} F_{\mu\nu} \right), \quad (7)$$

where  $F_{\alpha\beta}$  satisfies

$$\nabla_{\beta} F^{\alpha\beta} = 0, \quad \nabla_{[\alpha} F_{\beta\delta]} = 0. \quad (8)$$

We are interested in the self-force per unit mass on the particle at linear order in  $m$  and  $e$ . To make this precise, one could consider a family of solutions  $g_{\alpha\beta}(m, e), F_{\alpha\beta}(m, e)$  whose source for nonzero  $m$  and  $e$  is a body of finite extent, where  $e/m$  has a finite limit as  $m \rightarrow 0$  and where the characteristic spatial length of the body is, like  $e$ , linear in  $m$  for small  $m$ . At  $m = 0$ , the spacetime is the electrovac background, and the  $m \rightarrow 0$  limit of the family of trajectories is given by the Lorentz force law of that background,

$$a_{\alpha} = \frac{e}{m} F_{\alpha\beta} u^{\beta}, \quad (9)$$

where  $u^{\alpha}$  is the particle's velocity, and  $a^{\alpha} = u^{\beta} \nabla_{\beta} u^{\alpha}$  is its acceleration relative to the background geometry, and  $\nabla_{\alpha}$  is the covariant derivative of the background metric. The self-force arises from the perturbations in the gravitational and electromagnetic fields due to the body. We denote by  $\delta Q$  the linear perturbation in a quantity  $Q(m, e)$ ,

$$\delta Q := m \left. \frac{\partial}{\partial m} Q(m, e) \right|_{(m,e)=(0,0)} + e \left. \frac{\partial}{\partial e} Q(m, e) \right|_{(m,e)=(0,0)}. \quad (10)$$

Then  $Q(m, e) = Q + \delta Q + O(m^2, em, e^2)$ , where  $Q \equiv Q(0, 0)$ . The perturbations  $h_{\alpha\beta} = \delta g_{\alpha\beta}$  and  $\delta F_{\alpha\beta}$  are the linearized gravitational and electromagnetic fields of a point particle

with trajectory satisfying Eq. (9). In the problems that motivate this approximation, the background spacetime is nonradiative and the perturbations are the retarded fields  $h_{\alpha\beta}^{\text{ret}}$  and  $\delta F_{\alpha\beta}^{\text{ret}}$  of the particle, but the renormalization procedure is unrelated to these restrictions.

In the remainder of the paper, as in the previous paragraph, the symbols  $g_{\alpha\beta}$  and  $F_{\alpha\beta}$  will refer to the background metric and electromagnetic field, and all indices will be raised and lowered by the background metric.

We assume that, to linear order in the perturbed fields, the trajectory  $z(\tau)$  of the particle satisfies the renormalized Lorentz-force law equation associated with the perturbed metric  $g_{\alpha\beta} + h_{\alpha\beta}$  and electromagnetic field  $F_{\alpha\beta} + \delta F_{\alpha\beta}$ ,

$$mu^\beta(m, e)\nabla_\beta u_\alpha(m, e) = eu^\beta(m, e)F_{\alpha\beta} + f_\alpha^{\text{ren}} + O(m^2, em, e^2). \quad (11)$$

where  $f_\alpha^{\text{ren}}$  is obtained from the formal expression (2) for the self-force by angle average and mass renormalization, as in Eq. (5).

We will show that the renormalization of Eq. (5) is equivalent to separate renormalization of the electromagnetic and gravitational contributions to the self-force  $f_\alpha$ . It will then follow that in the mode-sum renormalization, there is no mixing of gravitational and electromagnetic parts: The renormalization is equivalent to subtracting (1) a singular expression  $f_\alpha^{\text{sing}} = f_\alpha^{\text{EM}, \text{sing}} + f_\alpha^{\text{GR}, \text{sing}}$ , where  $f_\alpha^{\text{EM}, \text{sing}}$  is the purely electromagnetic contribution from a point charge moving along an accelerated trajectory (with no perturbed gravitational field); and  $f_\alpha^{\text{GR}, \text{sing}}$  is the purely gravitational contribution from a point mass moving along the same trajectory that would arise if there were no perturbed electromagnetic field.

We consider the field in a convex normal neighborhood  $C$  of the event  $z(0)$ , denote by  $x$  any point of  $C$  and by  $\epsilon$  the length of the unique geodesic from  $z(0)$  to  $x$ . We choose  $\tau = 0$  at the position of the particle where we renormalize.

We work in a Lorenz gauge for each field: Introducing the trace-reversed metric perturbation

$$\gamma_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h^\delta_\delta \quad (12)$$

and a vector potential  $\delta A_\alpha$  for which  $\delta F_{\alpha\beta} = \nabla_\alpha \delta A_\beta - \nabla_\beta \delta A_\alpha$ , we have

$$\nabla^\beta \gamma_{\alpha\beta} = 0, \quad \nabla^\beta \delta A_\beta = 0. \quad (13)$$

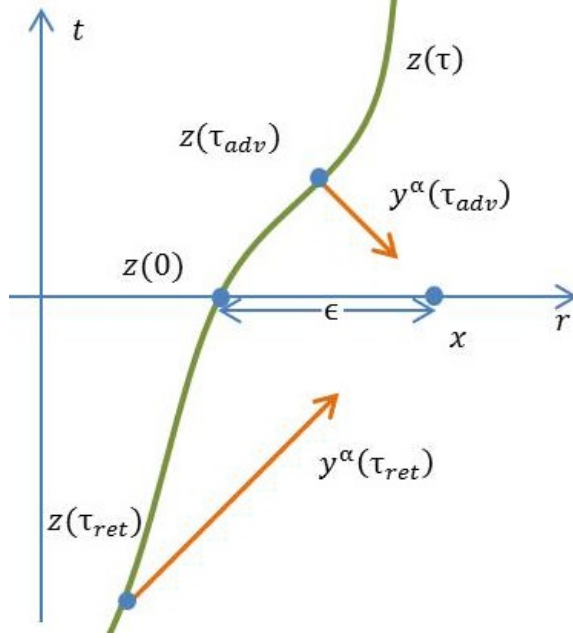


FIG. 1: The particle trajectory  $z(\tau)$ . Two null vectors  $y^\alpha(\tau_{ret})$  and  $y_\alpha(\tau_{adv})$  are tangent to future- and past-directed null geodesics from points along the trajectory to a field point  $x$ . A geodesic from  $z(0)$  to  $x$  has length  $\epsilon$ .

In this gauge, the perturbed Einstein equation,  $\delta G_{\alpha\beta} = 8\pi\delta T_{\alpha\beta}$ , has the form

$$\begin{aligned}
-2\delta G_{\alpha\beta} &= \square\gamma_{\alpha\beta} + 2\Omega_\alpha{}^\gamma{}_\beta{}^\delta\gamma_{\gamma\delta} \\
&= -16\pi m \int u_\alpha u_\beta \delta^{(4)}(x, z(\tau)) d\tau - 8 \left( F_{(\alpha}{}^\delta \delta_{\beta)}{}^\gamma - \frac{1}{4} g_{\alpha\beta} F^{\gamma\delta} \right) \delta F_{\gamma\delta} \\
&\quad + \left[ 4F_\alpha{}^\gamma F_\beta{}^\delta - 2F_{\alpha\epsilon} F_\beta{}^\epsilon g^{\gamma\delta} - g_{\alpha\beta} F^\gamma{}_\epsilon F^{\delta\epsilon} + F_{\epsilon\gamma} F^{\epsilon\gamma} \left( \delta_\alpha^\gamma \delta_\beta^\delta + \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \right) \right] \gamma_{\gamma\delta},
\end{aligned} \tag{14}$$

where  $\square = \nabla_\alpha \nabla^\alpha$  and

$$\Omega_\alpha{}^\gamma{}_\beta{}^\delta := R_{(\alpha}{}^\gamma{}_{\beta)}{}^\delta - R_{(\alpha}^\gamma \delta_{\beta)}^\delta - \frac{1}{2} g_{\alpha\beta} R^{\gamma\delta} + \frac{1}{2} R \delta_{(\alpha}^\gamma \delta_{\beta)}^\delta. \tag{15}$$

To make the notation more concise, we combine the last line of Eq. (14) with the term  $2\Omega_\alpha{}^\gamma{}_\beta{}^\delta\gamma_{\gamma\delta}$  to write

$$\square\gamma_{\alpha\beta} + 2\hat{\Omega}_\alpha{}^\gamma{}_\beta{}^\delta\gamma_{\gamma\delta} = -16\pi m \int u_\alpha u_\beta \delta^{(4)}(x, z(\tau)) d\tau - 16 \left( F_{(\alpha}{}^{[\delta} \delta_{\beta)}{}^{\gamma]} - \frac{1}{4} g_{\alpha\beta} F^{\gamma\delta} \right) \partial_\gamma \delta A_\delta, \tag{16}$$

with

$$\hat{\Omega}_\alpha{}^\gamma{}_\beta{}^\delta := \Omega_\alpha{}^\gamma{}_\beta{}^\delta - 2F_{(\alpha}{}^\gamma F_{\beta)}{}^\delta + F_{(\beta}{}^\epsilon F_{\alpha)\epsilon} g^{\gamma\delta} + g_{\alpha\beta} F^\gamma{}_\epsilon F^{\delta\epsilon} - \frac{1}{2} F_{\epsilon\gamma} F^{\epsilon\gamma} \left( \delta_{(\alpha}^\gamma \delta_{\beta)}^\delta - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \right). \tag{17}$$



The perturbed Maxwell equation,  $\delta(\nabla_\beta F^{\alpha\beta}) = 4\pi\delta j^\alpha$ , is given by

$$\square\delta A_\alpha - R_\alpha^\beta\delta A_\beta = -4\pi e \int u_\alpha\delta^{(4)}(x, z(\tau))d\tau - \nabla^\beta \left[ \left( F_{\beta}^{\gamma} \delta_\alpha^\delta + F_\alpha^\delta \delta_\beta^\gamma - \frac{1}{2}g^{\gamma\delta}F_{\alpha\beta} \right) \gamma_{\gamma\delta} \right]. \quad (18)$$

To find the singular behavior of the perturbed fields  $\gamma_{\alpha\beta}$  and  $\delta A_\alpha$ , we follow the formalism described in Paper I [15]. We introduce Riemann normal coordinates (RNCs)  $\{x^\mu\}$  with origin at  $z(0)$  and find the coordinate expansion of the perturbed fields. As in the case of particles with purely electromagnetic or gravitational interactions, the angle-average renormalization of Eq. (5) is equivalent to identifying and subtracting from  $f_\alpha^{ret}$  a singular part  $f_\alpha^{sing,\alpha}$ , for which the difference  $f_\alpha^{ret} - f_\alpha^{sing,\alpha}$  is continuous at the position of the particle. The singular expression  $f_\alpha^{sing}$  is in turn obtained from Eq. (2) by replacing  $\gamma_{\alpha\beta}$  and  $\delta A_\alpha$  by singular parts  $\gamma_{\alpha\beta}^{sing}$  and  $\delta A_\alpha^{sing}$  of the perturbed fields. As shown in Paper I, this prescription, inspired by the work of Gralla and Wald [11], agrees exactly with the Detweiler-Whiting singular field [22] when we apply it to scalar, or electric charges in any spacetime and for point masses in vacuum. A comparison with the singular potentials found by Poisson and Zimmerman [21] using the Detweiler-Whiting singular fields shows that shows that the angle-average renormalization is again equivalent to the renormalizing using the Detweiler-Whiting prescription for the renormalized Green's functions.

Our approach to expanding the perturbed fields is tailored to the mode-sum regularization methods pioneered by Barack and Ori [16]. We decompose the field perturbations into two pieces,  $\gamma_{\alpha\beta} = {}_I\gamma_{\alpha\beta} + {}_{II}\gamma_{\alpha\beta}$  and  $\delta A = {}_I A_\alpha + {}_{II}A_\alpha$ , satisfying

$$\square {}_I\gamma_{\alpha\beta} + 2\hat{\Omega}_\alpha^{\gamma\delta} {}_I\gamma_{\gamma\delta} = -16\pi m \int u_\alpha u_\beta \delta^{(4)}(x, z(\tau))d\tau, \quad (19)$$

$$\square {}_I A_\alpha - R_\alpha^\beta {}_I A_\beta = -4\pi e \int u_\alpha \delta^{(4)}(x, z(\tau))d\tau, \quad (20)$$

and

$$\square {}_{II}\gamma_{\alpha\beta} + 2\hat{\Omega}_\alpha^{\gamma\delta} {}_{II}\gamma_{\gamma\delta} = -16\Lambda_{\alpha\beta}^{\gamma\delta} \partial_\gamma \delta A_\delta, \quad (21)$$

$$\square {}_{II}A_\alpha - R_\alpha^\beta {}_{II}A_\beta = -2\nabla^\beta \left[ \Lambda_{\alpha\beta}^{\gamma\delta} \gamma_{\gamma\delta} \right], \quad (22)$$

where

$$\Lambda_{\alpha\beta}^{\gamma\delta} = F_{(\alpha}^{\gamma} \delta_{\beta)}^{\delta} - \frac{1}{4}g_{\alpha\beta} F^{\gamma\delta}. \quad (23)$$

At dominant order in  $\epsilon$  for each of the four pieces, this is the decomposition of Eq. (10):

$${}_I\gamma_{\alpha\beta} = m \left. \frac{\partial}{\partial m} \gamma_{\alpha\beta}(m, e) \right|_{(m,e)=(0,0)} [1 + O(\epsilon)], \quad (24a)$$

$${}_{II}\gamma_{\alpha\beta} = e \left. \frac{\partial}{\partial e} \gamma_{\alpha\beta}(m, e) \right|_{(m,e)=(0,0)} [1 + O(\epsilon)], \quad (24b)$$

$${}_IA_\alpha = e \left. \frac{\partial}{\partial e} A_\alpha(m, e) \right|_{(m,e)=(0,0)} [1 + O(\epsilon)], \quad (24c)$$

$${}_{II}A_\alpha = m \left. \frac{\partial}{\partial m} A_\alpha(m, e) \right|_{(m,e)=(0,0)} [1 + O(\epsilon)]. \quad (24d)$$

We can quickly find the short-distance (Hadamard) expansion of the solutions to Eqs. (19) and (20), because their forms are nearly identical, respectively, to the equations governing the gravitational perturbation due to a massive particle with no charge, and to the electromagnetic perturbation due to a charged particle whose gravitational perturbation can be neglected. Eq. (20) is in fact the electromagnetic perturbation equation of a spacetime with no background electromagnetic field, but with the present background metric; its formal solutions are reviewed by Poisson *et al.* [2] and by the present authors in Paper I. Eq. (19) differs from the equation governing the metric perturbation of a point mass in a vacuum spacetime only by the substitution  $R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta} \rightarrow \hat{\Omega}_{\alpha}{}^{\gamma}{}_{\beta}{}^{\delta}$ . As seen in Appendix I, the Hadamard expansion of the field  ${}_I\gamma_{\alpha\beta}$  differs only by the same substitution from the formal expansion found in Paper I for accelerated motion in a vacuum background.

The RNC expansions of solutions  ${}_I\gamma_{\alpha\beta}$  and  ${}_IA_\alpha$  to Eqs. (19) and (20) are given by

$$\begin{aligned} \frac{1}{m} {}_I\gamma_{\alpha\beta} = & \frac{4u_\alpha u_\beta - 8u_{(\alpha} a_{\beta)} u_\gamma x^\gamma}{\sqrt{S}} + 4u_\mu u_\nu \hat{\Omega}_{(\alpha}{}^{\mu}{}_{\beta)}{}^{\nu} \sqrt{S} \\ & + \frac{4x^\mu x^\nu}{\sqrt{S}} \left[ (a_\alpha a_\beta + \dot{a}_{(\alpha} u_{\beta)})(q_{\mu\nu} + u_\mu u_\nu) + 2a_{(\alpha} u_{\beta)} a_\mu u_\nu - \frac{u_{(\alpha} R_{\beta)\epsilon\gamma\sigma} u^\gamma}{3} (\delta^\epsilon{}_\mu \delta^\sigma{}_\nu + u^\epsilon \delta^\sigma{}_\mu u_\nu) \right] \\ & + O(\epsilon^2). \end{aligned} \quad (25)$$

and

$$\begin{aligned} \frac{1}{e} {}_IA_\alpha = & \frac{u_\alpha - a_\alpha u_\beta x^\beta}{\sqrt{S}} + \frac{(u_\alpha R_{\gamma\delta} - 2u^\beta R_{\alpha(\gamma\delta)\beta})}{12\sqrt{S}} [\delta^\gamma{}_\mu \delta^\delta{}_\nu + u^\gamma u^\delta (q_{\mu\nu} + u_\mu u_\nu) + 2u^\gamma \delta^\delta{}_\nu u_\mu] x^\mu x^\nu \\ & + \frac{[2a_\alpha u_\mu a_\nu + \dot{a}_\alpha (q_{\mu\nu} + u_\mu u_\nu)] x^\mu x^\nu}{2\sqrt{S}} + \frac{6R_{\alpha\beta} u^\beta - u_\alpha R}{12} \sqrt{S} + O(\epsilon^2), \end{aligned} \quad (26)$$

where  $2S = q_{\alpha\beta}y_{ret}^\alpha y_{ret}^\beta + q_{\alpha\beta}y_{adv}^\alpha y_{adv}^\beta$ , where the vector  $y^\alpha$  is tangent to the unique null geodesic from the trajectory at  $z(\tau_{ret})$  or  $z(\tau_{adv})$  to the field point  $x$ .

In Eqs. (21) and (22) for  ${}_{II}\gamma_{\alpha\beta}$  and  ${}_{II}A_\alpha$ , the left sides involve the same linear operators as those of Eqs. (19) and (20). The right sides are constructed not only from the solutions we have just obtained for  ${}_I\gamma_{\alpha\beta}$  and  ${}_IA_\alpha$  but also from the fields  ${}_{II}\gamma_{\alpha\beta}$  and  ${}_{II}A_\alpha$  themselves. We can obtain local solutions iteratively, noting that each solution is higher order in  $\epsilon$  than its source. In particular, the leading terms in  ${}_I\gamma_{\alpha\beta}$  and  ${}_IA_\alpha$  proportional to  $1/\sqrt{S_0}$  give dominant terms in  ${}_{II}\gamma_{\alpha\beta}$  and  ${}_{II}A_\alpha$  of subleading order,  $O(\epsilon^0)$ . The first iteration then uses on the right side the leading terms in  ${}_I\gamma_{\alpha\beta}$  and  ${}_IA_\alpha$ :

$$\square {}_{II}\gamma_{\alpha\beta} + O(\epsilon^0) = -16e\Lambda_{\alpha\beta}{}^{\gamma\delta}u_\delta|_{x=z(0)}\partial_\gamma\left(\frac{1}{\sqrt{S_0}}\right) \quad (27)$$

$$\square {}_{II}A_\alpha + O(\epsilon^0) = -8m\Lambda_{\gamma\delta\alpha}{}^\beta u^\gamma u^\delta|_{x=z(0)}\partial_\beta\left(\frac{1}{\sqrt{S_0}}\right), \quad (28)$$

where  $S_0 = q_{\alpha\beta}x^\alpha x^\beta$ .

Solving Eqs. (28) and (27) as RNC expansions, we find

$$\begin{aligned} {}_{II}\gamma_{\alpha\beta} &= -8e\frac{u_\delta\Lambda_{\alpha\beta}{}^{\gamma\delta}q_{\gamma\epsilon}x^\epsilon}{\sqrt{S_0}} + O(\epsilon) \\ &= -2m\frac{x^\gamma}{\sqrt{S_0}}\left(2a_{(\alpha}\eta_{\beta)\gamma} - 2\frac{e}{m}u_{(\beta}F_{\alpha)\gamma} - \eta_{\alpha\beta}a_\gamma\right) + O(\epsilon), \end{aligned} \quad (29)$$

$$\begin{aligned} {}_{II}A_\alpha &= -4m\frac{u_\gamma u_\delta\Lambda_{\alpha\beta}{}^{\gamma\delta}q_\epsilon^\beta x^\epsilon}{\sqrt{S_0}} + O(\epsilon) \\ &= -\frac{m}{\sqrt{S_0}}\left[F_{\alpha\beta} + \frac{m}{e}(a_\alpha u_\beta - 2u_\alpha a_\beta)\right]x^\beta + O(\epsilon). \end{aligned} \quad (30)$$

Here and from now on, when the symbols  $a_\alpha$ ,  $u^\alpha$ ,  $q_{\alpha\beta}$  and  $F_{\alpha\beta}$  appear without explicit  $x$  dependence, they denote the values of the corresponding quantities at the position  $z(0)$  of the particle.

This first iteration is already enough for our principal results: The singular part of the self-force at leading and subleading order and, in particular, its contribution to the renormalized mass are unchanged by the gravitational-electromagnetic coupling. The result is due to a remarkable cancellation of the contributions to the self force from the two mixed terms. That is, the contributions proportional to  $em$  in the electromagnetic and gravitational parts of the self-force are equal and opposite. To see this, we compute the force using

$$\begin{aligned} f_\alpha^{EM} &= e\delta F_{\alpha\beta}u^\beta = (\delta_\alpha^\beta u^\delta - \delta_\alpha^\delta u^\beta)\partial_\beta\delta A_\delta, \\ f_\alpha^{GR} &= -mq_\alpha^\delta(\nabla_\beta h_{\gamma\delta} - \frac{1}{2}\nabla_\delta h_{\beta\gamma})u^\beta u^\gamma = \frac{m}{4}[q_\alpha^\beta(q^{\gamma\delta} + u^\gamma u^\delta) - 4q_\alpha^\delta u^\beta u^\gamma]\nabla_\beta\gamma_{\gamma\delta}. \end{aligned} \quad (31)$$

Substituting  ${}_{II}\gamma_{\alpha\beta}$  and  ${}_{II}A_\alpha$  in Eqs. (29) and (30) gives the contributions proportional to  $em$ , namely

$$\begin{aligned} {}_{II}f^{EM}_\alpha &= -emu^\beta F_{\gamma\beta} \left( \frac{\delta_\alpha^\gamma}{\sqrt{S_0}} - \frac{q_{\alpha\delta}x^\delta x^\gamma}{S_0^{3/2}} \right) + O(\epsilon^0) \\ &= -{}_{II}f^{GR}_\alpha. \end{aligned} \quad (32)$$

Note that the angle-average of each contribution,

$$\langle {}_{II}f^{s=1,2}_\alpha \rangle = \mp \frac{2}{3} em F_{\alpha\beta} u^\beta \frac{1}{\sqrt{S_0}} = \mp \frac{2}{3} \frac{m^2}{\sqrt{S_0}} a_\alpha, \quad (33)$$

is proportional to  $a_\alpha$  and would contribute to the mass renormalization if the terms did not cancel.

The sums  ${}_I\gamma_{\alpha\beta} + {}_{II}\gamma_{\alpha\beta}$  and  ${}_I\delta A_\alpha + {}_{II}\delta A_\alpha$  are the singular fields to  $O(\epsilon^0)$ :

$$\gamma_{\alpha\beta}^{sing} = 4 \frac{mu_\alpha u_\beta - 2 \left( ma_{(\alpha} u_{\beta)} u_\epsilon + eu_\delta \Lambda_{\alpha\beta}^{\gamma\delta} q_{\gamma\epsilon} \right) x^\epsilon}{\sqrt{S}} + O(\epsilon), \quad (34)$$

$$\delta A_\alpha^{sing} = \frac{eu_\alpha - \left( ea_\alpha u_\epsilon + 4mu_\gamma u_\delta \Lambda_{\alpha\beta}^{\gamma\delta} q_\epsilon^\beta \right) x^\epsilon}{\sqrt{S}} + O(\epsilon). \quad (35)$$

We will now continue the iteration to obtain an  $O(\epsilon)$  contribution  ${}_{II}\gamma_{\alpha\beta}$  and  ${}_{II}A_\alpha$  by including on the right side of Eqs. (21) and (22) their known expansions through  $O(\epsilon^0)$ . We obtain in this way the  $O(\epsilon)$  contribution to the singular fields  $\gamma_{\alpha\beta}^{sing}$  and  $\delta A_\alpha^{sing}$  up to a homogeneous solution to the flat-space wave equation of the form  $P^{(2n)}(x)/S_0^{n-1/2}$ , where  $P^{(2n)}$  is a homogeneous polynomial of degree  $2n$  in the coordinates  $\{x^\mu\}$ . Substituting the expressions (34) and (35) for  $\gamma_{\alpha\beta}$  and  $\delta A_\alpha$  back into Eqs. (21) and (22) respectively, we have

$$\square {}_{II}\gamma_{\alpha\beta} + 2\hat{\Omega}_\alpha^{\gamma\delta} {}_{II}\gamma_{\gamma\delta} = -16\Lambda_{\alpha\beta}^{\gamma\delta}(x) \partial_\gamma \left( \frac{eu_\delta + A_{\delta\epsilon} x^\epsilon + O(\epsilon^2)}{\sqrt{S}} \right), \quad (36)$$

$$\square {}_{II}A_\alpha - R_{\alpha}^{\beta} {}_{II}A_\beta = -\nabla^\beta \left[ \Lambda_{\alpha\beta}^{\gamma\delta}(x) \left( \frac{8mu_\gamma u_\delta + 2\gamma_{\gamma\delta\epsilon} x^\epsilon + O(\epsilon^2)}{\sqrt{S}} \right) \gamma_{\gamma\delta} \right]. \quad (37)$$

where  $A_{\alpha\beta}$  and  $\gamma_{\alpha\beta\gamma}$  are defined by

$$A_{\alpha\beta} := -ea_\alpha u_\beta - 4m\Lambda_{\gamma\delta\alpha\epsilon} u^\gamma u^\delta q_\beta^\epsilon, \quad (38)$$

$$\gamma_{\alpha\beta\gamma} := -8 \left( ma_{(\alpha} u_{\beta)} u_\gamma + e\Lambda_{\alpha\beta\delta\epsilon} q_\gamma^\delta u^\epsilon \right). \quad (39)$$

The RNC expansion of  $\Lambda_{\gamma\delta}^{\beta\alpha}(x)$  about  $z(0)$  is given by

$$\Lambda_{\gamma\delta}^{\beta\alpha}(x) = \Lambda_{\gamma\delta}^{\beta\alpha}|_{x=z(0)} + \Lambda_{\gamma\delta}^{\beta\alpha}{}_\epsilon x^\epsilon + O(\epsilon^2), \quad (40)$$

where

$$\Lambda_{\gamma\delta}^{\beta\alpha} = \partial_\epsilon \Lambda_{\gamma\delta}^{\beta\alpha}|_{x=z(0)} = \left( \partial_\epsilon F_{(\alpha}^{[\delta} \delta_{\beta)}^{\gamma]} - \frac{1}{4} \eta_{\alpha\beta} \partial_\epsilon F^{\gamma\delta} \right)_{x=z(0)}. \quad (41)$$

Solving Eqs. (36) and (37) for  ${}_{II}\gamma_{\alpha\beta}$  and  ${}_{II}A_\alpha$  to  $O(\epsilon)$  and adding the result to the expansions of  ${}_I\gamma_{\alpha\beta}$  and  ${}_IA_\alpha$ , we obtain the singular fields to sub-subleading order, namely

$$\begin{aligned} \gamma_{\alpha\beta}^{local} &= \frac{4mu_\alpha u_\beta + \gamma_{\alpha\beta\epsilon} x^\epsilon}{\sqrt{S}} + \frac{4mx^\gamma x^\delta [(a_\alpha a_\beta + \dot{a}_{(\alpha} u_{\beta)}) (q_{\gamma\delta} + u_\gamma u_\delta) + 2a_{(\alpha} u_{\beta)} a_\gamma u_\delta]}{\sqrt{S}} \\ &- \frac{4m}{3} \frac{u_{(\alpha} R_{\beta)\epsilon} \gamma_\delta q_\lambda^\epsilon u^\gamma x^\delta x^\lambda}{\sqrt{S}} + 4mu_\gamma u_\delta \hat{\Omega}_\alpha^{\gamma\delta} \sqrt{S} + 4\Lambda_{\alpha\beta}^{\gamma\delta} A_{\delta\epsilon} (u^\epsilon u_\lambda - \delta_\lambda^\epsilon) \left( \delta_\gamma^\lambda \sqrt{S} + \frac{q_{\gamma\mu} x^\mu x^\lambda}{\sqrt{S}} \right) \\ &+ 4eu_\delta \Lambda_{\alpha\beta}^{\gamma\delta} \left[ (u^\epsilon u_\lambda - \delta_\lambda^\epsilon) \frac{q_{\gamma\mu} x^\lambda x^\mu}{\sqrt{S}} + q_\gamma^\epsilon \sqrt{S} \right] + O(\epsilon^2), \end{aligned} \quad (42)$$

and

$$\begin{aligned} \delta A_\alpha^{local} &= \frac{eu_\alpha + A_{\alpha\beta} x^\beta}{\sqrt{S}} + e \left( \frac{u_\alpha R_{\gamma\delta} - 2u^\beta R_{\alpha(\gamma\delta)\beta}}{12\sqrt{S}} \right) [\delta_\mu^\gamma \delta_\nu^\delta + 2u^\gamma u_\mu \delta_\nu^\delta + u^\gamma u^\delta (q_{\mu\nu} + u_\mu u_\nu)] x^\mu x^\nu \\ &+ \frac{e}{2} \left[ \frac{2a_\alpha a_\delta u_\gamma + \dot{a}_\alpha (q_{\gamma\delta} + u_\gamma u_\delta)}{\sqrt{S}} \right] x^\gamma x^\delta + \frac{e}{2} u^\beta R_{\alpha\beta} \sqrt{S} + 4m \frac{\Lambda_{\gamma\delta\alpha\beta} u^\beta u^\gamma u^\delta u_\mu a_\nu x^\mu x^\nu}{\sqrt{S}} \\ &+ \frac{4mu_\gamma u_\delta \Lambda_{\alpha\beta\epsilon}^{\gamma\delta} + \Lambda_{\alpha\beta}^{\gamma\delta} \gamma_{\gamma\delta\epsilon}}{2} (u^\epsilon u_\lambda - \delta_\lambda^\epsilon) \left( \eta^{\lambda\beta} \sqrt{S} + \frac{q_\mu^\beta x^\lambda x^\mu}{\sqrt{S}} \right) + O(\epsilon^2). \end{aligned} \quad (43)$$

The fields  $\gamma_{\alpha\beta}^{local}$  and  $\delta A_{\alpha\beta}^{local}$  coincide with the singular fields at leading and subleading order. We could now impose conditions on the Green's function analogous to those of Eqs. (A4) and (A5) to complete the sub-subleading part of the singular field, and from it compute the sub-subleading piece of the singular force, but we are saved the trouble: As in Eqs. (42) and (43) the sub-subleading terms are functions even in the coordinates  $x^\mu$ . Because the expressions for the self-force in Eqs. (31) are proportional to the gradients of the potentials, they are odd in  $x^\mu$  and will therefore vanish upon angle averaging. The remaining contributions to the self-force are at leading and subleading order,  $O(\epsilon^{-2})$  and  $O(\epsilon^{-1})$ , and we have

$$\begin{aligned} f_\alpha^{sing} &= (e^2 - m^2) \left[ \frac{q_{\alpha\beta} x^\beta}{S_0^{3/2}} - [q_{\alpha\beta} a_\gamma (3\eta_{\epsilon\delta} - 2q_{\epsilon\delta}) - a_\alpha q_{\gamma\delta} \eta_{\epsilon\beta}] \frac{x^\gamma x^\delta x^\beta x^\epsilon}{S_0^{5/2}} \right] \\ &+ \frac{(4m^2 - e^2) a_\alpha}{\sqrt{S_0}} + O(\epsilon^0). \end{aligned} \quad (44)$$

The term  $O(\epsilon^0)$  can be written as a seventh order polynomial in  $x^\mu$  divided by  $S_0^{7/2}$ , manifestly odd in the RNCs. This implies not only their angle average vanishes, but also that they do not contribute to the regularization parameters in mode-sum regularization.

With the cancellation of the mixed terms (terms proportional to  $em$ ) in the expression for the self-force,  $f_\alpha^{sing}$  at subleading order is unaltered by the coupling of the electromagnetic and gravitational fields when it is written in terms of  $g_{\alpha\beta}$ ,  $u_\alpha$ ,  $a_\alpha$  and the RNCs. A charge  $e$  moving with this acceleration in a geometry with this metric but with no background electromagnetic field has  $f_\alpha^{sing}$  given by the part of the present  $f_\alpha^{sing}$  that is proportional to  $e^2$ ; and, as shown in Paper I, a mass  $m$ , again moving on the same accelerated trajectory but with non-gravitational interactions ignored, has an  $f_\alpha^{sing}$  given by the terms proportional to  $m^2$ . As we discuss in the next section only the leading and subleading regularization parameters  $A_\alpha$  and  $B_\alpha$  are required for mode-sum regularization, and they are each simply the sum of similarly independent electromagnetic and gravitational parameters.

In Appendix B, using the potentials obtained by Zimmerman and Poisson for a particle of scalar charge  $q$  and mass  $m$  moving in a scalarvac spacetime, we find that the analogous result holds. Again to subleading order, there is no mixed contribution to the singular expression for the self-force;  $f_\alpha^{sing}$  is at this order the sum of its purely gravitational and scalar terms; and the mode-sum regularization requires only parameters  $A_\alpha$  and  $B_\alpha$  that are each the sum of independent gravitational and electromagnetic parameters.

### III. MODE-SUM REGULARIZATION

In mode-sum regularization, one decomposes retarded and singular fields in angular harmonics associated with a given set of spherical coordinates. The method is commonly used for black-hole spacetimes with Schwarzschild or Boyer-Lindquist coordinates, for example, but the formalism is valid for arbitrary spherical coordinates  $(\hat{x}^\mu) = (t, r, \theta, \phi)$  related in the usual way to a smooth Cartesian chart  $(t, x^1, x^2, x^3)$ , for which 2-spheres of constant  $t$  and  $r$  are in the domain of the chart. We choose  $t = 0$  at the position  $z(0)$  of the particle and denote its coordinates by  $(0, r_0, \theta_0, \phi_0)$ . Using the angular harmonic decomposition of the field perturbations, one similarly decomposes  $f_\alpha^{ret}$  and  $f_\alpha^{sing}$  as sums of angular harmonics on a sphere through the particle and writes the renormalized self-force as the convergent sum

$$f_\alpha^{ren} = \sum_{\ell=0}^{\infty} (f_\alpha^{ret,\ell} - f_\alpha^{sing,\ell}), \quad (45)$$

where

$$f_{\alpha}^{ret/sing,\ell} = \lim_{r \rightarrow r_0} \sum_{\mathfrak{m}=-\ell}^{\ell} \int d\Omega f_{\alpha}^{ret/sing}(0, r, \theta_0, \phi_0) \bar{Y}_{\ell\mathfrak{m}}(\theta, \phi). \quad (46)$$

(Here  $f_{\alpha}^{ret,\ell}$  and hence  $f_{\alpha}^{sing,\ell}$  have leading terms whose sign depends on whether the  $r \rightarrow 0$  limit is taken from  $r < r_0$  or  $r > r_0$ .) Because  $f_{\alpha}^{sing}$  is defined only in a normal neighborhood of  $z(0)$  one first arbitrarily extends it to a thickened 2-sphere, and  $f_{\alpha}^{sing}$  then refers to this smooth extension. As mentioned in Sect. I, Paper I generalized to accelerated motion in generic spacetimes a result found by Barack and Ori [16] for geodesic motion in Schwarzschild and Kerr geometries, that  $f_{\alpha}^{sing,\ell}$  has the form  $A_{\alpha}L + B_{\alpha} + O(L^{-2})$ , where  $L = \ell + 1/2$ ,  $A_{\alpha}$  and  $B_{\alpha}$  are independent of  $\ell$ , and the convergent sum of the  $O(L^{-2})$  part of  $f_{\alpha}^{sing,\ell}$  vanishes in a Lorenz gauge.

The singular fields,  $\gamma_{\alpha\beta}^{sing}$  and  $\delta A_{\alpha}^{sing}$ , differ at subleading order from the values computed in Paper I for accelerated motion of an electric charge  $e$  and of a point mass  $m$ . The difference in the expression for the singular part of the self-force, however, coincides at subleading order, with the sum of the electromagnetic and gravitational expressions in Paper I. As in Paper I, the terms of this order have the form of a polynomial of odd degree in the RNCs, divided by a half-integral power of  $S_0$ , and that fact, together with the form of the leading and subleading terms, implies Eq. (6). We then have the result

$$f_{\alpha}^{ren} = \sum_{\ell=0}^{\infty} [f_{\alpha}^{ret,\ell} - (A_{\alpha}L + B_{\alpha})], \quad (47)$$

where the values of  $A_{\alpha}$  and  $B_{\alpha}$  are just the sums

$$A_{\alpha} = A_{\alpha}^{GR} + A_{\alpha}^{EM}, \quad B_{\alpha} = B_{\alpha}^{GR} + B_{\alpha}^{EM}, \quad (48)$$

of the gravitational and electromagnetic parameters obtained in Paper I from the perturbed gravitational and electromagnetic fields of accelerated charges and masses associated with a choice  $(t, r, \theta, \phi)$  of spherical coordinates. These are given in Eqs. (C20) and (C21) of Paper I. We display the explicit values of  $A_{\alpha}$  and  $B_{\alpha}$  in Appendix C.

#### IV. CONCLUSIONS

With the assumption that one can renormalize the self-force on a charged point mass moving in an electrovac spacetime by a combination of angle-average and mass renormalization, we show that the renormalization can be done as if the equations for the perturbed

electromagnetic and gravitational fields were decoupled. We showed that, as in the case of uncoupled fields, the renormalized self-force can be obtained by a mode-sum regularization that involves only the leading and subleading regularization parameters  $A_\alpha$  and  $B_\alpha$ ; and each of these parameters is a sum of their previously calculated values for purely electromagnetic and gravitational perturbations from an accelerated point charge and a point mass, respectively. The renormalization we obtain is also equivalent to that found in independent work by Zimmerman and Poisson. The perturbed fields are, of course, coupled, and the renormalized self-force has mixed terms, terms proportional to  $em$  as well as to  $e^2$  and  $m^2$ .

What has not yet been done is to extend to the coupled fields we consider here the underlying justification based on variants of matched asymptotic expansion for electromagnetic or gravitational interactions alone. The equivalence of the several renormalization prescriptions that we and Poisson and Zimmerman have considered, however, gives us confidence that no surprise will emerge.

Finally, the results obtained here may be useful in studying the problem of overcharging a black hole, with charged particles moving in a background Reissner-Nordstrom spacetime.

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## Appendix A: Gravitational Green's function in a non-vacuum spacetime

We will make extensive use of the treatment found in [2]. The goal is to find the Green's function  $G^{\alpha\beta}_{\gamma'\delta'}(x, x')$ , where  $x$  and  $x'$  are two arbitrary points in a convex normal neighborhood  $C$ , and unprimed and primed indices are tensor indices at  $x$  and  $x'$ , respectively. When we apply this to solve for  ${}_I\gamma_{\alpha\beta}$  in Eq. (25), we set  $x' = z(0)$ . We consider the purely



gravitational Green's function, the solution to

$$\square G^{\alpha\beta}_{\gamma'\delta'}(x, x') + 2\hat{\Omega}^{\alpha\beta}_{\mu\nu} G^{\mu\nu}_{\gamma'\delta'}(x, x') = -4\pi g^{(\alpha}_{\gamma'}(x, x') g^{\beta)}_{\delta'}(x, x') \delta^{(4)}(x, x'), \quad (\text{A1})$$

where  $g^{\alpha}_{\gamma'}(x, x')$  is the bivector of parallel transport, taking a vector,  $v^{\gamma'}(x')$ , defined at  $x'$  and parallel transporting it along the unique geodesic connecting  $x$  and  $x'$ , resulting in  $v^{\alpha}(x, x') = g^{\alpha}_{\gamma'} v^{\gamma'}(x')$ .

The retarded and advanced Green's functions  $G^{\alpha\beta}_{\gamma'\delta'\pm}(x, x')$  have the form,

$$G^{\alpha\beta}_{\gamma'\delta'\pm}(x, x') = U^{\alpha\beta}_{\gamma'\delta'}(x, x') \delta_{\pm}(\sigma) + V^{\alpha\beta}_{\gamma'\delta'}(x, x') \theta_{\pm}(-\sigma), \quad (\text{A2})$$

where the distributions  $\delta_{\pm}$  and  $\theta_{\pm}$  are defined in Section 13 of [2], and  $\sigma$  is Synge's world function. Substituting Eq. (A2) into the left hand side of Eq. (A1), we find (with the argument  $(x, x')$  of bitensors suppressed)

$$\begin{aligned} \square G^{\alpha\beta}_{\gamma'\delta'} + 2\hat{\Omega}^{\alpha\beta}_{\mu\nu} G^{\mu\nu}_{\gamma'\delta'} &= -4\pi U^{\alpha\beta}_{\gamma'\delta'} \delta^{(4)}(x, x') + \delta'_{\pm}(\sigma) \left( 2U^{\alpha\beta}_{\gamma'\delta';\gamma} \sigma^{\gamma} + (\sigma^{\gamma}_{\gamma} - 4) U^{\alpha\beta}_{\gamma'\delta'} \right) \\ &\quad + \delta_{\pm}(\sigma) \left( -2V^{\alpha\beta}_{\gamma'\delta';\gamma} \sigma^{\gamma} + (2 - \sigma^{\gamma}_{\gamma}) V^{\alpha\beta}_{\gamma'\delta'} + (\square U^{\alpha\beta}_{\gamma'\delta'} + 2\hat{\Omega}^{\alpha\beta}_{\mu\nu} U^{\mu\nu}_{\gamma'\delta'}) \right) \\ &\quad + \theta_{\pm}(-\sigma) \left( \square V^{\alpha\beta}_{\gamma'\delta'} + 2\hat{\Omega}^{\alpha\beta}_{\mu\nu} V^{\mu\nu}_{\gamma'\delta'} \right) \\ &= -4\pi g^{(\alpha}_{\gamma'} g^{\beta)}_{\delta'} \delta^{(4)}(x, x'). \end{aligned} \quad (\text{A3})$$

In comparing this to the corresponding (unnumbered) equation in [2] (between Eq. 16.7 and 16.8), it is clear that the only difference is that the tensor  $R^{\alpha}_{\gamma}{}^{\beta}_{\delta}$  is replaced here by  $\hat{\Omega}^{\alpha}_{\gamma}{}^{\beta}_{\delta}$ . Following the same technique used in [2], we require that the coefficients of  $\delta'_{\pm}(\sigma)$ ,  $\delta_{\pm}(\sigma)$ , and  $\theta_{\pm}(\sigma)$  separately vanish. We thereby find,

$$\begin{aligned} U^{\alpha\beta}_{\gamma'\delta'}(x, x') &= g^{(\alpha}_{\gamma'}(x, x') g^{\beta)}_{\delta'}(x, x') \Delta^{1/2}(x, x') \\ &= g^{(\alpha}_{\gamma'}(x, x') g^{\beta)}_{\delta'}(x, x') \left( 1 + \frac{1}{12} R_{\gamma'\delta'} \sigma^{\gamma'} \sigma^{\delta'} + O(\epsilon^3) \right), \end{aligned} \quad (\text{A4})$$

and

$$V^{\alpha\beta}_{\gamma'\delta'}(x', x') = \frac{\delta^{(\alpha'}_{\gamma'} \delta^{\beta')_{\delta'}} R(x')}{12} + \hat{\Omega}^{\alpha'}_{(\gamma'}{}^{\beta'}_{\delta')}(x'). \quad (\text{A5})$$

Since  $R = 0$  for electrovac, the only difference between the Hadamard expansions of a point mass in vacuum and  $I\gamma_{\alpha\beta}$  is in the bitensor  $V^{\alpha\beta}_{\gamma'\delta'}$ , where instead of the Riemann tensor we have  $\hat{\Omega}^{\alpha'}_{(\gamma'}{}^{\beta'}_{\delta')}$ .

## Appendix B: Decoupling in renormalization of a massive scalar charge.

Using work on a massive scalar charge by Zimmerman and Poisson [21] (ZP), we verify here that there is no cross-term at subleading order in the singular expression for the self-force of a massive particle with scalar charge moving in a background scalarvac spacetime. The result implies that, as in the case of a point charge in an electrovac spacetime, the renormalized mass is obtained by subtracting (1) the scalar-field contribution from a point charge moving along an accelerated trajectory and (2) the purely gravitational contribution from a point mass moving along the same trajectory. In a mode-sum regularization, the regularization parameters are then sums of their purely scalar and gravitational values.

Subleading terms in  $f_\alpha^{sing}$  proportional to  $q\mathbf{m}$  arise from terms of order  $\epsilon^0$  in  $\Phi^{sing}$  that are proportional to  $m$  and from terms of order  $\epsilon^0$  in  $\gamma_{\alpha\beta}^{sing}$  that are proportional to  $q$ . We consider first the contribution to the self-force from  $\Phi^{sing}$ . From Eq. (6.19) of [21], written in terms of our RNCs with origin at  $z(0)$ , we have

$$\Phi^{sing} = \frac{1}{\sqrt{S_0}}[\gamma_1 U + u_\alpha x^\alpha \gamma_2 \dot{U} + O(\epsilon^2)], \quad (\text{B1})$$

where  $\gamma_1$  and  $\gamma_2$  are independent of the perturbed fields, with  $\gamma_1[z(0)] = \gamma_2[z(0)] = 1$ . From Eq. (7.25) of ZP,  $U$  and  $\dot{U}$  have no terms proportional to  $\mathbf{m}$ , implying that there is no  $q\mathbf{m}$  contribution to  $f_\alpha^{sing}$  from  $\Phi^{sing}$  at subleading order,  $O(\epsilon^{-1})$ .

We turn next to the contribution from  $\gamma_{\alpha\beta}^{sing}$ . The symbol  $\hat{r}$  in ZP is  $\hat{r} = u_\epsilon x^\epsilon$ . Again from Eq. (6.19),

$$\gamma_{sing}^{\alpha\beta} = \frac{1}{\rho}[\gamma_1 U^{\alpha\beta} + u_\epsilon x^\epsilon \gamma_2 \dot{U}^{\alpha\beta} + O(\epsilon^2)],$$

with (7.25) giving no term in  $U^{\alpha\beta}$  proportional to  $q$  and, in  $\dot{U}^{\alpha\beta}$ , the single term

$$q \frac{\partial}{\partial q} \dot{U}^{\alpha\beta} = -4q \dot{\Phi} u^\alpha u^\beta.$$

From (6.21),  $\gamma_2 = 1 + O(\epsilon)$ , and the single term proportional to  $q$  in  $\gamma^{\alpha\beta}$  is then

$$q \frac{\partial}{\partial q} \gamma^{\alpha\beta} = -4 \frac{u_\gamma x^\gamma}{\sqrt{S_0}} q \dot{\Phi} u^\alpha u^\beta.$$

The contribution of this term to the self-force at subleading order is then

$$\frac{m}{4} \left[ q_i^\beta (q^{\gamma\delta} + u^\gamma u^\delta) - 4q_i^\gamma u^\beta u^\delta \right] \nabla_\beta \gamma_{\gamma\delta} \Big|_{t=0} = mq \dot{\Phi} \left[ \frac{u_\beta x^\beta q_{\alpha\gamma} x^\gamma}{S_0^{3/2}} \right]_{t=0} = 0, \quad (\text{B2})$$

using  $u_i = 0$ . We conclude that there is no contribution to the self force through subleading order that is proportional to  $qm$ .

### Appendix C: Integrals for the B term

We give here the explicit forms of the regularization parameters  $A_\alpha$  and  $B_\alpha$  associated with arbitrary spherical coordinates  $t, r, \theta, \phi$ , with  $(t_0, r_0, \theta_0, \phi_0)$  the coordinates of the position of the particle. The value of  $A_\alpha$  is obtained as a limit  $r \rightarrow r_0$  from  $r < r_0$  or  $r > r_0$ , and it is given by

$$A_{\alpha\pm} = \mp(e^2 - m^2) \sin \theta_0 \frac{q_{\alpha r} - q_{\alpha\theta}q_{\theta r}/q_{\theta\theta} - q_{\alpha\phi}q_{\phi r}/q_{\phi\phi}}{(q_{\phi\phi}q_{\theta\theta}q_{rr} - q_{\phi\phi}q_{\theta r}^2 - q_{\theta\theta}q_{\phi r}^2)^{1/2}}. \quad (C1)$$

In writing the components of  $B_\alpha$ , it is helpful to define a tensor  $c_{\beta\gamma}^\alpha$  at  $z(0)$ , whose only nonvanishing components are  $c_{\phi\phi}^\theta = 4^{-1} \sin(2\theta_0)$  and  $c_{\theta\phi}^\phi = c_{\phi\theta}^\phi = -2^{-1} \cot \theta_0$ . Then

$$B_\alpha = \frac{P_{\alpha\beta\gamma\delta\epsilon}}{2\pi} I^{\beta\gamma\delta\epsilon}, \quad (C2)$$

where

$$\begin{aligned} P_{\alpha\beta\gamma\delta\epsilon} = & (e^2 - m^2) [q_{\alpha\beta}a_\gamma(3g_{\epsilon\delta} - 2q_{\epsilon\delta}) - a_\alpha q_{\gamma\delta}g_{\epsilon\beta} + (3q_{\beta\lambda}q_{\alpha\epsilon} - q_{\alpha\lambda}q_{\beta\epsilon})c_{\gamma\delta}^\lambda] \\ & + (4m^2 - e^2)a_\alpha q_{\beta\gamma}q_{\delta\epsilon} \end{aligned} \quad (C3)$$

and the components  $I^{\alpha\beta\gamma\delta}$  are zero unless all indices are either  $\theta$  or  $\phi$ . The nonzero values of  $I^{\alpha\beta\gamma\delta}$  are all proportional to functions of the quantities

$$\alpha = \sin^2 \theta_0 \, q_{\theta\theta}/q_{\phi\phi} - 1 \text{ and } \beta = 2 \sin \theta_0 \, q_{\theta\phi}/q_{\phi\phi}. \quad (C4)$$

To display them, we follow Barack [1], defining an integer  $N$ ,  $0 \leq N \leq 4$ , whose value is the number of times the index  $\phi$  occurs in the set  $\{\alpha, \beta, \gamma, \delta\}$ :  $N = \delta_\alpha^\phi + \delta_\beta^\phi + \delta_\gamma^\phi + \delta_\delta^\phi$ . Then, when each index of  $I^{\alpha\beta\gamma\delta}$  is  $\theta$  or  $\phi$ ,

$$I^{\alpha\beta\gamma\delta} = \frac{(\sin \theta_0)^{5-N}}{(\alpha^2 + \beta^2)^2(4\alpha + 4 - \beta^2)^{3/2}(Q/2)^{1/2}} \left[ Q I_K^{(N)} \hat{K}(\omega) + I_E^{(N)} \hat{E}(\omega) \right], \quad (C5)$$

where

$$Q = \alpha + 2 - (\alpha^2 + \beta^2)^{1/2} \text{ and } \omega = \frac{2(\alpha^2 + \beta^2)^{1/2}}{\alpha + 2 + (\alpha^2 + \beta^2)^{1/2}}. \quad (C6)$$

The various integrals for the  $B$  term, computed first in [23] are given below:

$$\begin{aligned} I_K^{(0)} &= 4 [12\alpha^3 + \alpha^2(8 - 3\beta^2) - 4\alpha\beta^2 + \beta^2(\beta^2 - 8)], \\ I_E^{(0)} &= -16 [8\alpha^3 + \alpha^2(4 - 7\beta^2) + \alpha\beta^2(\beta^2 - 4) - \beta^2(\beta^2 + 4)], \end{aligned} \quad (C7)$$

$$I_K^{(1)} = 8\beta [9\alpha^2 - 2\alpha(\beta^2 - 4) + \beta^2] ,$$

$$I_E^{(1)} = -4\beta [12\alpha^3 - \alpha^2(\beta^2 - 52) + \alpha(32 - 12\beta^2) + \beta^2(3\beta^2 + 4)] , \quad (C8)$$

$$I_K^{(2)} = -4 [8\alpha^3 - \alpha^2(\beta^2 - 8) - 8\alpha\beta^2 + \beta^2(3\beta^2 - 8)] ,$$

$$I_E^{(2)} = 8 [4\alpha^4 + \alpha^3(\beta^2 + 12) + \alpha(\beta^2 - 4)(3\beta^2 - 2\alpha) + 2\beta^2(3\beta^2 - 4)] , \quad (C9)$$

$$I_K^{(3)} = 8\beta [\alpha^3 - 7\alpha^2 + \alpha(3\beta^2 - 8) + \beta^2] ,$$

$$I_E^{(3)} = -4\beta [8\alpha^4 - 4\alpha^3 + \alpha^2(15\beta^2 - 44) + 4\alpha(5\beta^2 - 8) + \beta^2(3\beta^2 + 4)] , \quad (C10)$$

$$I_K^{(4)} = -4 [4\alpha^4 - 4\alpha^3 + \alpha^2(7\beta^2 - 8) + 12\alpha\beta^2 - \beta^2(\beta^2 - 8)] ,$$

$$I_E^{(4)} = 16 [4\alpha^5 + 4\alpha^4 + \alpha^3(7\beta^2 - 4) + \alpha^2(11\beta^2 - 4) + (2\alpha + 1)\beta^2(\beta^2 + 4)] . \quad (C11)$$

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